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Constrained Solution of a System of Matrix Equations

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Abstract

This report presents a technique for constrained solution of a system of matrix equations which arises in the problem of pole placement with static dissipative output feedback. Previously developed necessary conditions for the existence of a solution are shown to be sufficient as well. A minimax approach is presented to determine a feasible coefficient vector that satisfies these conditions. A procedure to construct the desired solution matrix, based on minimax programming techniques, is detailed. Numerical examples are presented to illustrate the application of this approach.

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Constrained Solution of a System of Matrix Equations

Introduction

A technique for constrained solutions of the system of matrix equations

$$\begin{aligned} GW_1p &= V_1p \\ GW_2p &= V_2p \end{aligned} \tag{1}$$

where $W_i, \quad V_i, \quad i = 1, 2,$ are given $m \times n$ data matrices, G is an $m \times m$ unknown matrix, and p is an arbitrary unknown $n \times 1$ coefficient vector, is described in this report. The solution matrix, G, is constrained such that its symmetric part, that is, $\mathrm{sym}\{G\} \triangleq \frac{1}{2}\big(G + G^T\big)$, is positive semidefinite. A further constraint on the solution is also considered, wherein the solution matrix G is symmetric and positive semidefinite. Such systems of matrix equations arise in the eigensystem assignment problem with dissipativity constraints[1,2]. The eigenpair assignment problem is reduced to a system of equations in Eq. (1), where G is the unknown feedback gain matrix that must satisfy the dissipativity constraints and p is an arbitrary coefficient vector. Specifically, Eq. (14) of Ref. 2 corresponds to Eq. (1) above, which is investigated in this report. Necessary conditions on the arbitrary coefficient vector, p, for the existence of a solution matrix, G, are available in Ref. 2.

This report shows that the necessary conditions on the arbitrary coefficient vector, p, for the existence of a solution matrix[2] are sufficient as well. This development reduces the problem of constrained solution of the system of matrices to determining a feasible coefficient vector. A minimax approach to determine a coefficient vector satisfying these conditions is presented. The technique to obtain a matrix, G, that satisfies the system of equations in Eq. (1) along with the constraints of positive semidefiniteness is detailed. A numerical example has been presented to illustrate this technique.

Solution Technique

First, a proposition presents necessary and sufficient conditions on the arbitrary coefficient vector, p, for the existence of a constrained solution to the system of matrices in Eq. (1).

Proposition. A matrix G whose symmetric part is positive semidefinite satisfies Eq. (1) if and only if there exists a vector p which satisfies

$$\begin{gathered} p^T V_1^T W_1 p \geq 0 \\ p^T V_2^T W_2 p \geq 0 \\ \left(p^T V_1^T W_1 p\right)\left(p^T V_2^T W_2 p\right) - \frac{1}{4}\left(p^T V_1^T W_2 p + p^T V_2^T W_1 p\right)^2 \geq 0 \end{gathered} \tag{2}$$

Furthermore, a symmetric and positive semidefinite matrix G solves Eq. (1) if and only if there exists a vector p which satisfies the inequalities in Eq. (2) and $p^T V_1^T W_2 p = p^T V_2^T W_1 p$.

Proof: The necessity of the conditions in Eq. (2) for existence of a solution has been proved in Ref. 2. The following presents another approach to this proof. Denote $y_1 = V_1 p,\ y_2 = V_2 p,\ x_1 = W_1 p$ and $x_2 = W_2 p$. Then, the conditions in Eq. (2) become $y_1^T x_1 \geq 0,\quad y_2^T x_2 \geq 0$ and $\left(y_1^T x_1\right)\left(y_2^T x_2\right) - \frac{1}{4}\left(y_1^T x_2 + y_2^T x_1\right)^2 \geq 0$. Eq. (1) can be written as $G[x_1 \quad x_2] = [y_1 \quad y_2]$, and premultiplying this by $[x_1 \quad x_2]^T$ leads to

$$\begin{aligned} [x_1 \quad x_2]^T G [x_1 \quad x_2] &= [x_1 \quad x_2]^T [y_1 \quad y_2] \\ &= \begin{bmatrix} x_1^T y_1 & x_1^T y_2 \\ x_2^T y_1 & x_2^T y_2 \end{bmatrix} \end{aligned} \tag{3}$$

Adding Eq. (3) and its transpose yields

$$[x_1 \quad x_2]^T [\mathrm{sym}(G)] [x_1 \quad x_2] = \begin{bmatrix} y_1^T x_1 & 0.5\left(y_1^T x_2 + y_2^T x_1\right) \\ 0.5\left(y_1^T x_2 + y_2^T x_1\right) & y_2^T x_2 \end{bmatrix} \tag{4}$$

Now if $\mathrm{sym}(G) \geq 0$, then Eq. (4) implies that

$$\begin{bmatrix} y_1^T x_1 & 0.5\left(y_1^T x_2 + y_2^T x_1\right) \\ 0.5\left(y_1^T x_2 + y_2^T x_1\right) & y_2^T x_2 \end{bmatrix} \geq 0 \tag{5}$$

Since determinants of the principal minors of a positive semidefinite matrix must be nonnegative, the conditions in Eq. (2) follow. Thus, if there exists a matrix G, which satisfies Eq. (1), and its symmetric part is positive semidefinite, then the vector p must satisfy conditions in Eq. (2).

For sufficiency, it has to be shown that if a vector p satisfies the conditions in Eq. (2), then there exists a matrix G, whose symmetric part is positive semidefinite, which satisfies Eq. (1). A constructive proof of this statement follows. Denote $X = [x_1 \quad x_2]$ and $Y = [y_1 \quad y_2]$. Then, Eq. (1) may be rewritten as $GX = Y$. Let Q be an orthogonal matrix, such that

$$Q^T Y = \begin{bmatrix} \widetilde{Y}_1 \\ 0 \end{bmatrix} \tag{6}$$

where $\widetilde{Y}_1$ is a nonsingular 2×2 matrix. The matrix Q can be obtained by QR factorization[3] of Y. Define $\widetilde{X}_1$, $\widetilde{X}_2$ as follows

$$\begin{bmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \end{bmatrix} = Q^T X \tag{7}$$

where $\widetilde{X}_1$ is a 2×2 matrix, and $\widetilde{X}_2$ is a $(m-2) \times 2$ matrix. Note that $\widetilde{X}_1$ is nonsingular if x_1 and x_2 are linearly independent (otherwise, Eq. (1) is solved trivially). Defining $\widetilde{G}_{11} = \widetilde{Y}_1 \widetilde{X}_1^{-1}$, it can be seen that

$$\begin{bmatrix} \widetilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{Y}_1 \\ 0 \end{bmatrix} \tag{8}$$

Therefore, it follows that the matrix G defined as

$$G = Q \begin{bmatrix} \widetilde{G}_{11} & 0 \\ 0 & 0 \end{bmatrix} Q^T \tag{9}$$

satisfies $GX = Y$.

Next, it is shown that $\mathrm{sym}(G) \geq 0$. From Eq. (5) it follows that conditions in Eq. (2) imply $\mathrm{sym}\left(Y^T X\right) \geq 0$. Next, since $Y^T X = \left(Q^T Y\right)^T \left(Q^T X\right) = \widetilde{Y}_1^T \widetilde{X}_1$, $\mathrm{sym}\left(Y^T X\right) \geq 0$ leads to $\mathrm{sym}\left(\widetilde{Y}_1^T \widetilde{X}_1\right) \geq 0$. Noting that $\mathrm{sym}\left(\widetilde{Y}_1^T \widetilde{X}_1\right) = \widetilde{X}_1^T \left[\mathrm{sym}\left(\widetilde{G}_{11}\right)\right] \widetilde{X}_1 \geq 0$,

$\mathrm{sym}\left(\widetilde{G}_{11}\right) \geq 0$ follows from the nonsingularity of $\widetilde{X}_1$. Finally, by construction, $\mathrm{sym}(G) \geq 0$ if $\mathrm{sym}\left(\widetilde{G}_{11}\right) \geq 0$.

Furthermore, if G is symmetric, that is, $G = \mathrm{sym}(G)$, then $y_1^T x_2 = x_1^T G^T x_2 = x_2^T G^T x_1 = y_2^T x_1$, so that the additional condition in the proposition is satisfied. On the other hand, the additional constraint ensures that $Y^T X = \widetilde{Y}_1^T \widetilde{X}_1$ is symmetric. Since $\widetilde{Y}_1 = \widetilde{G}_{11} \widetilde{X}_1$, $\widetilde{Y}_1^T \widetilde{X}_1 = \widetilde{X}_1^T \widetilde{Y}_1$, and $\widetilde{X}_1$ is nonsingluar, it follows that $\widetilde{G}_{11}$ is symmetric. Finally, again by construction, G is symmetric when $\widetilde{G}_{11}$ is symmetric. ■

The next step in solution of Eq. (1) is to determine a coefficient vector, p, that satisfies conditions of the proposition, Eq. (2). A number of approaches have been attempted for this problem[2]. A very efficient and reliable numerical approach to determining a feasible coefficient vector, based on minimax optimization, is presented below.

The conditions of Eq. (2) may be used directly for a minimax optimization. However, note that while the first two conditions form a quadratic in the coefficient vector, p, the last condition is much more complicated, potentially leading to numerical inefficiencies in the optimization algorithms. Sufficient conditions on the coefficient vector p, in terms of four quadratic inequalities, are as follows:

$$
\begin{aligned}
f_1(p) &= p^T \left\{ V_1^T W_1 + \frac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right) \right\} p \geq 0 \\
f_2(p) &= p^T \left\{ V_1^T W_1 - \frac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right) \right\} p \geq 0 \\
f_3(p) &= p^T \left\{ V_2^T W_2 + \frac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right) \right\} p \geq 0 \\
f_4(p) &= p^T \left\{ V_2^T W_2 - \frac{1}{2}\left(V_1^T W_2 + V_2^T W_1\right) \right\} p \geq 0
\end{aligned}
\tag{10}
$$

The conditions together imply that the coefficient vector, p satisfies

$$
\begin{aligned}
p^T V_1^T W_1 p - \left| \frac{1}{2}\left(p^T V_1^T W_2 p + p^T V_2^T W_1 p\right) \right| \geq 0 \\
p^T V_2^T W_2 p - \left| \frac{1}{2}\left(p^T V_1^T W_2 p + p^T V_2^T W_1 p\right) \right| \geq 0
\end{aligned}
\tag{11}
$$

It can be readily verified that if the coefficient vector p satisfies the conditions in Eq. (11), then the inequalities of Eq. (2) are also satisfied.

Since the four expressions on the left hand side of the inequalities in Eq. (10) are quadratic in the coefficient vector, p, a feasible coefficient vector can be reliably determined using numerical programming techniques. The approach is to maximize the minimum of the four quadratic functions, $f_i(p),\ i = 1, ..., 4$, with respect to p, until all of them are positive. This problem becomes a standard minimax problem by reversing the sign of the quadratic functions, that is, minimizing the maximum of $-f_i(p),\ i = 1, ..., 4$ with respect to p. By introducing a scalar variable, λ, the minimax problem is transformed to a constrained minimization problem, as follows:

$$\min_{p,\lambda} \lambda \qquad \text{such that } f_i(p) + \lambda \geq 0,\ i = 1, ..., 4 \tag{12}$$

Standard nonlinear programming techniques may be used for this constrained minimization[4]. Analytic gradients of the quadratic functions, $f_i(p),\ i = 1, ..., 4$, are readily available, since the gradient of any quadratic function is given as $\frac{\partial}{\partial p}\left(p^T Q p\right) = \left(Q + Q^T\right)p$. The optimization in Eq. (12) is very well-behaved because the functions, $f_i(p),\ i = 1, ..., 4$, are quadratic in p, and analytic gradients are linear in p. A nonpositive value of λ in Eq. (12) provides a feasible coefficient vector, p, that satisfies the conditions in Eq. (2). The convergence of the nonlinear optimization in Eq. (12) is not an issue, since the search can be terminated once a desirable (negative) target value of λ has been attained.

Furthermore, if a symmetric and positive semidefinite solution of Eq. (1) is desired, it follows from the Proposition that an additional equality constraint,

$$g(p) = p^T\left(V_1^T W_2 - V_2^T W_1\right)p = 0 \tag{13}$$

must be satisfied apart from the conditions in Eq. (2). It is noted that this equality constraint again involves a quadratic in p, so that its analytic gradient is linear and readily available. Therefore, to obtain a feasible coefficient vector, p, for a symmetric and positive semidefinite solution matrix, G, the equality constraint in Eq. (13) must be included with the optimization of Eq. (12). Again, a feasible vector p is obtained as soon as the scalar parameter, λ, attains a nonpositive value.

Experience in application of the minimax approach presented above has shown that this technique is very effective in obtaining a feasible coefficient vector, p, which satisfies the conditions in Eq. (2).

Once a feasible coefficient vector, p, satisfying the conditions of the Proposition has been obtained, a constrained solution of Eq. (1) may be constructed as follows:

1. Form the $m \times 2$ matrices, $Y = [V_1 p \quad V_2 p]$, and $X = [W_1 p \quad W_2 p]$
2. Perform a QR decomposition of Y to obtain $\widetilde{Y}_1$ as in Eq. (6), and obtain $\widetilde{X}_1$ from Eq. (7).
3. Using $\widetilde{G}_{11} = \widetilde{Y}_1 \widetilde{X}_1^{-1}$, form the solution matrix G as in Eq. (9).

From the proof of the Proposition, it follows that starting with a feasible coefficient vector, the matrix, G, constructed from the steps above is a desired solution of the system of matrix equations in Eq. (1).

Numerical Example

A numerical example is presented in this section to demonstrate the solution technique presented in this report. The data matrices (W_1, W_2, V_1, V_2) used in this example have been obtained from the numerical example in Ref. 2 for damping enhancement of a model of a flexible space structure at NASA Langley. For this example, $m = 4$ and $n = 8$. The 4×8 data matrices W_1, W_2, V_1, V_2 are given below.

$$V_1 = \begin{bmatrix} 0.002 & -0.202 & 0.326 & -0.468 & -0.063 & 0.001 & -0.002 & 0.005 \\ 0.000 & 0.721 & -0.406 & -0.156 & 0.005 & 0.028 & 0.042 & 0.015 \\ -0.002 & -0.338 & 0.279 & -0.063 & 0.055 & 0.035 & 0.051 & 0.023 \\ 0.001 & -0.205 & -0.167 & 0.809 & -0.031 & 0.026 & 0.044 & 0.010 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 0.063 & -0.001 & 0.002 & -0.005 & 0.002 & -0.202 & 0.326 & -0.468 \\ -0.005 & -0.028 & -0.042 & -0.015 & 0.000 & 0.721 & -0.406 & -0.156 \\ -0.055 & -0.035 & -0.051 & -0.023 & -0.002 & -0.338 & 0.279 & -0.063 \\ 0.031 & -0.026 & -0.044 & -0.010 & 0.001 & -0.205 & -0.167 & 0.809 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0.267 & 0.092 & 0.137 & 0.041 & -0.790 & 0.175 & 0.259 & 0.095 \\ 0.000 & 0.073 & 0.116 & 0.028 & -0.131 & 0.256 & 0.370 & 0.143 \\ -0.238 & 0.051 & 0.091 & 0.016 & 0.465 & 0.299 & 0.454 & 0.186 \\ 0.137 & 0.068 & 0.105 & 0.038 & -0.446 & 0.156 & 0.250 & 0.170 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0.790 & -0.175 & -0.259 & -0.095 & 0.267 & 0.092 & 0.137 & 0.041 \\ 0.131 & -0.256 & -0.370 & -0.143 & 0.000 & 0.073 & 0.116 & 0.028 \\ -0.465 & -0.299 & -0.454 & -0.186 & -0.238 & 0.051 & 0.091 & 0.016 \\ 0.446 & -0.156 & -0.250 & -0.170 & 0.137 & 0.068 & 0.105 & 0.038 \end{bmatrix}$$

The problem is to determine a 4×4 matrix G whose symmetric part is positive semidefinite and solves Eq. (1) for some 8×1 coefficient vector p.

The first step is to determine a feasible coefficient vector, p, which satisfies the conditions of the Proposition. This is done by solving the nonlinear optimization problem in Eq. (12). Note that upper and lower bounds have to be imposed on the elements of the coefficient vector p for solution of this optimization problem. The upper bounds were set to 1.0, and the lower bounds were set to -1.0 for the current solution. Optimization software of Refs. 5 and 6 is used to determine a feasible value of p as

$$p = [1.000 \quad 1.000 \quad 1.000 \quad 1.000 \quad -0.966 \quad 1.000 \quad -0.404 \quad 0.948\,]^T$$

Using this feasible value of the coefficient vector, p, the first step is to form the matrices Y and X, which are

$$Y = \begin{bmatrix} -0.275 & -0.721 \\ 0.180 & 0.647 \\ -0.140 & -0.674 \\ 0.485 & 0.579 \end{bmatrix} \qquad X = \begin{bmatrix} 1.462 & 0.078 \\ 0.587 & -0.585 \\ -0.239 & -1.146 \\ 0.995 & -0.202 \end{bmatrix}$$

For the second step, a QR decomposition[5] of Y results in

$$Q = \begin{bmatrix} -0.456 & 0.308 & -0.237 & 0.801 \\ 0.299 & -0.473 & 0.630 & 0.538 \\ -0.233 & 0.631 & 0.722 & -0.162 \\ 0.805 & 0.532 & -0.159 & 0.207 \end{bmatrix}$$

and $\widetilde{Y}_1 = \begin{bmatrix} 0.602 & 1.146 \\ 0.000 & -0.645 \end{bmatrix}$. Using Eq. (7), it follows that $\widetilde{X}_1 = \begin{bmatrix} 0.366 & -0.106 \\ 0.551 & -0.530 \end{bmatrix}$.

Finally, $\widetilde{G}_{11} = \widetilde{Y}_1 \widetilde{X}_1^{-1} = \begin{bmatrix} 7.019 & -3.565 \\ -2.624 & 1.741 \end{bmatrix}$, and from Eq. (9), a constrained solution of Eq. (1) is

$$G = \begin{bmatrix} 2.494 & -2.220 & 2.298 & -2.078 \\ -2.104 & 1.890 & -1.969 & 1.683 \\ 2.095 & -1.895 & 1.983 & -1.623 \\ -2.539 & 2.191 & -2.218 & 2.392 \end{bmatrix}$$

Eigenvalues of the symmetric part of G are $\{8.447,\ 0.3133,\ 0.000,\ 0.000\}$, which shows that it is positive semidefinite. It can be readily checked that this matrix satisfies Eq. (1).

Similarly, for a symmetric and positive semidefinite solution, the additional equality constraint of Eq. (13) is included in the optimization problem, to obtain a feasible coefficient vector as

$$p = [0.074 \quad -0.176 \quad -0.244 \quad -0.085 \quad 0.071 \quad -0.081 \quad -0.136 \quad -0.068\,]^T$$

Following the steps above, a symmetric and positive semidefinite solution of the system of equations is obtained as

$$G = \frac{1}{10} \begin{bmatrix} 0.352 & 0.312 & -0.245 & 0.165 \\ 0.312 & 0.947 & 0.467 & 0.092 \\ -0.245 & 0.467 & 0.867 & -0.169 \\ 0.165 & 0.092 & -0.169 & 0.082 \end{bmatrix}$$

Eigenvalues of this matrix are $\{0.138,\ 0.087,\ 0.000,\ 0.000\}$, which shows that this matrix is symmetric and positive definite. Again, it can be readily verified that this matrix does satisfy Eq. (1).

This solution technique has been used for various other data sets corresponding to different problems of damping enhancement of flexible space structures. It has been found to be very efficient and reliable on all problems attempted thus far.

Summary

This report has presented an approach to the constrained solution of a system of matrix equations which arises frequently in pole placement with output feedback under dissipativity constraints. It has been shown that previously available necessary conditions for the existence of constrained solutions are sufficient as well. A minimax approach to determine a feasible coefficient vector satisfying the conditions for existence of a solution was presented, and the steps to construct a constrained solution to the system of matrix equations have been described. This approach has been successfully applied to the design of static dissipative controllers for eigensystem assignment in several flexible structure applications, and has proven to be very reliable and efficient.

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